

Chapter 7 Iterative methods

$$\bar{x} = \{x_1, \dots, x_n\}^T$$

$$l_2 \text{ norm of } \bar{x} \text{ is defined as } \|\bar{x}\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$$

$$l_\infty \text{ norm of } \bar{x} \text{ is defined as } \|\bar{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

$$\|\bar{x} - \bar{y}\|_2 = \sqrt{(\bar{x} - \bar{y})^T (\bar{x} - \bar{y})}$$

$$\text{Natural matrix norm } \|A\| = \max_{\|\bar{x}\|=1} \|A\bar{x}\|, \quad A\bar{x} = \left\{ \sum_{j=1}^n a_{ij} x_j \right\}$$

$$l_2 \text{ norm of matrix } A \text{ is } \|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|$$

$$l_\infty \text{ norm of } A \text{ is } \|A\|_2 = \max_{\|\bar{x}\|_2=1} \|A\bar{x}\|_2$$

$$\text{If } \{\bar{x}^{(k)}\}_{k=1} \rightarrow \bar{x}^* \text{ or } \lim_{k \rightarrow \infty} x_i^{(k)} = x_i^* \text{ or } \lim_{k \rightarrow \infty} \|\bar{x}^{(k)} - \bar{x}^*\| = 0$$

then $\{\bar{x}^{(k)}\}_{k=1}$ is convergent to $\bar{x}^*$.

The eigenvalue λ and its corresponding eigenvector $\bar{x}$ of the matrix A is $A\bar{x} = \lambda\bar{x}$.

The spectral radius $\rho(A)$ of a matrix A is defined by

$$\rho(A) = \max |\lambda|.$$

Properties:

$$1. \|A\|_2 = [\rho(A^T A)]^{1/2}$$

$$2. \rho(A) \text{ for any natural norm}$$

$$A \text{ is a convergent matrix iff } \lim_{n \rightarrow \infty} \|A^n\| = 0, \text{ or } \rho(A) < 1 \text{ or } \lim_{n \rightarrow \infty} A^n \bar{x} = \bar{0}$$

The problem $A\bar{x} = \bar{b}$ can be expressed as $\bar{x} = T\bar{x} + \bar{c}$

If $\rho(T) < 1$, then the iteration $\bar{x}^{(k+1)} = T\bar{x}^{(k)} + \bar{c}$ converges.

$$\bar{x}^{(k+1)} = T(T\bar{x}^{(k-1)} + \bar{c}) + \bar{c} = T^2\bar{x}^{(k-1)} + (T + I)\bar{c}$$

$$= T^k\bar{x}^{(0)} + (T^{k-1} + \dots + T^0)\bar{c}$$

$$\lim_{k \rightarrow \infty} T^k = O, \quad S_k = T^{k-1} + \dots + T^0, \quad TS_k = T^k + \dots + T,$$

$$S_k - TS_k = (I - T)S_k = I - T^k, \quad S_k = (I - T)^{-1} - (I - T)^{-1}T^k$$

$$\lim_{k \rightarrow \infty} S_k = (I - T)^{-1}, \quad \lim_{k \rightarrow \infty} \bar{x}^{(k+1)} = (I - T)^{-1}\bar{c}.$$

Jacobi iteration

Express $A = D + L + U$, then $D\bar{x} = -(L + U)\bar{x} + \bar{b}$ or

$$\bar{x} = -D^{-1}(L + U)\bar{x} + D^{-1}\bar{b} \equiv T\bar{x} + \bar{c}$$

$$x_i^{(k+1)} = -\sum_{j \neq i} \frac{a_{ij}}{a_{ii}} x_j^{(k+1)} + \frac{b_i}{a_{ii}}$$

Gauss-Seidel method

$$(D + L)\bar{x} = -U\bar{x} + \bar{b} \quad \text{or} \quad \bar{x} = -(D + L)^{-1}U\bar{x} + (D + L)^{-1}\bar{b} \equiv T_g\bar{x} + \bar{c}$$

SOR (successive over-relaxation) method

$$D\bar{x} + \omega(D + L + U)\bar{x} = D\bar{x} + \omega\bar{b}, \quad D\bar{x} + \omega(D + L)\bar{x} = D\bar{x} - \omega U\bar{x} + \omega\bar{b}$$

$$(D + \omega L)\bar{x} = [(1 - \omega)D - \omega U]\bar{x} + \omega\bar{b}$$

$$\bar{x} = (D + \omega L)^{-1}[(1 - \omega)D - \omega U]\bar{x} + \omega(D + \omega L)^{-1}\bar{b} \equiv T_\omega\bar{x} + \bar{c}$$

Iterative Refinement

Suppose $\bar{x}^{(0)}$ to be an approximate solution of $A\bar{x}^{(0)} = \bar{b}$.

The error $\bar{b}^{(1)} = \bar{b} - A\bar{x}^{(0)}$. Let $\bar{x}^{(1)}$ be an approximate solution of

$A\bar{x}^{(1)} = \bar{b}^{(1)}$. Then the solution of $A\bar{x} = \bar{b}$ is $\bar{x} = \bar{x}^{(0)} + \bar{x}^{(1)}$.

§ The conjugate gradient method

The solution $\bar{x}$ of $A\bar{x} = \bar{b}$ is the optimal solution of

$$\min_{\bar{x}} g(\bar{x}) = \min_{\bar{x}} \left(\frac{1}{2} \bar{x}^T A \bar{x} - \bar{x}^T \bar{b} \right), \quad A = A^T.$$

Define $h(t) = g(\bar{x} + t\bar{v})$

How to find t and $\bar{v}$ such that $g(\bar{x} + t\bar{v}) \leq g(\bar{x})$

$$\begin{aligned} h(t) &= \frac{1}{2} (\bar{x} + t\bar{v})^T A (\bar{x} + t\bar{v}) - (\bar{x} + t\bar{v})^T \bar{b} \\ &= \left(\frac{1}{2} \bar{x}^T A \bar{x} - \bar{x}^T \bar{b} \right) + t\bar{v}^T (A\bar{x} - \bar{b}) + \frac{1}{2} t^2 \bar{v}^T A \bar{v} \\ &= g(\bar{x}) + t\bar{v}^T (A\bar{x} - \bar{b}) + \frac{1}{2} t^2 \bar{v}^T A \bar{v} \end{aligned}$$

$$0 = h'(\hat{t}) = \bar{v}^T (A\bar{x} - \bar{b}) + \hat{t} \bar{v}^T A \bar{v} \quad \text{or}$$

$$\hat{t} = -\bar{v}^T (A\bar{x} - \bar{b}) / \bar{v}^T A \bar{v} = (\bar{v}, b - A\bar{x}) / (\bar{v}, A\bar{v})$$

$$h(\hat{t}) = g(\bar{x}) - \frac{1}{2} \frac{(\bar{v}, A\bar{x} - \bar{b})^2}{(\bar{v}, A\bar{v})} \leq g(\bar{x})$$

Given $\bar{x}^{(1)}$

$$\bar{v}^{(k)} = b - A\bar{x}^{(k)}, \quad t_k = (\bar{v}^{(k)}, \bar{v}^{(k)}) / (\bar{v}^{(k)}, A\bar{v}^{(k)}),$$

$$\bar{x}^{(k+1)} = \bar{x}^{(k)} + t_k \bar{v}^{(k)}.$$

$$= g(\bar{x}) + t\bar{v}^T (A\bar{x} - \bar{b}) + \frac{1}{2} t^2 \bar{v}^T A \bar{v}$$

$$0 = h'(\hat{t}) = \bar{v}^T (A\bar{x} - \bar{b}) + \hat{t} \bar{v}^T A \bar{v} \quad \text{or}$$

$$\hat{t} = -\bar{v}^T (A\bar{x} - \bar{b}) / \bar{v}^T A \bar{v} = (\bar{v}, b - A\bar{x}) / (\bar{v}, A\bar{v})$$

Given $\bar{x}^{(1)}$ and $\bar{v}^{(1)}$

$$t_k = (\bar{v}^{(k)}, b - A\bar{x}^{(k)}) / (\bar{v}^{(k)}, A\bar{v}^{(k)}), \quad \bar{x}^{(k)} = \bar{x}^{(k-1)} + t_k \bar{v}^{(k)}, \quad \bar{v}^{(k+1)} = b - A\bar{x}^{(k)}$$

Ex. The vector $\bar{x}^* = \{2 \ 1\}^T$ is the root of

$$A\bar{x} = \bar{b}, \quad A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \bar{b} = \begin{Bmatrix} 5 \\ 4 \end{Bmatrix}$$

$$\text{Try } \bar{x}^{(0)} = \{1 \ 1\}^T, \quad \bar{v}^{(1)} = \begin{Bmatrix} 5 \\ 4 \end{Bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix}, \quad \bar{v}^{(1)T} \bar{v}^{(1)} = 5, \quad \bar{v}^{(1)} A \bar{v} = 14$$

$$t_1 = \frac{\bar{\mathbf{v}}^T \bar{\mathbf{v}}}{\bar{\mathbf{v}}^T A \bar{\mathbf{v}}} = \frac{5}{14}, \quad \bar{\mathbf{x}}^{(1)} = \bar{\mathbf{x}}^{(0)} + t_1 \bar{\mathbf{v}}^{(1)} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \frac{5}{14} \begin{Bmatrix} 2 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1 + \frac{10}{14} \\ 1 + \frac{5}{14} \end{Bmatrix}$$

$$\left\| \bar{\mathbf{x}}^* - \bar{\mathbf{x}}^{(1)} \right\| = \frac{\sqrt{17}}{14} < 1 = \left\| \bar{\mathbf{x}}^* - \bar{\mathbf{x}}^{(0)} \right\|$$

HW7

Solve the problem

$$4x_1 - x_2 - x_4 = 0$$

$$-x_1 + 4x_2 - x_3 - x_5 = -1$$

$$-x_2 + 4x_3 + x_5 - x_6 = 9$$

$$-x_1 + 4x_4 - x_5 - x_6 = 4$$

$$-x_2 - x_4 + 4x_5 - x_6 = 8$$

$$-x_3 - x_5 + 4x_6 = 6$$

by (a) Jacobi method, (b) Gauss-Seidel method, (c) SOR method, and (d) the conjugate gradient method.